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Factorisation of Bäcklund transformations

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Abstract. The factorisation of the L operator in a Lax pair $\{L, P\}$ has played an important role in the theory of integrable systems. For example, the Miura transformation can be immediately obtained from the factorisation of the Schrödinger operator; $-\partial_x^2 + u - k^2 = (-\partial_x + a)(\partial_x + b)$ requires that $u = k^2 - a_x + a^2$ and $a = b$. For zero curvature representations this procedure is not available, however. In this paper we attempt to produce an analogous theory of factorisations for zero-curvature representations.

1. Factorisation of zero-curvature representations

Consider the Zakharov-Shabat-Mikhailov zero-curvature representation of a solvable non-linear equation [1, 2]:

$$Y_x = P(k)Y \qquad Y_t = Q(k)Y \tag{1.1}$$

where $P(k)$ and $Q(k)$ are matrix-valued functionals of the variables in the solvable equation and which also depend upon a spectral parameter k. Then, as is well known, provided (1.1) is isospectral ($k_t = 0$) the condition of integrability of (1.1) is equivalent to the associated solvable equation being satisfied. Since the solvable equations admit auto-Bäcklund transformations (ABT) we can represent the equation by $\{P, Q\}$ where $(P_i, Q_i) \in \{P, Q\}$ satisfies the zero-curvature equations of type (1.1) with the fundamental solution Y_i.

The auto-Bäcklund transformations are defined by k-dependent gauge transformations of (1.1). If we start with the solution (P_0, Q_0) then define

$$Y_1(k) = T(k)Y_0(k) \tag{1.2}$$

and the new solution $(P_1, Q_1) \in \{P, Q\}$ is given by

$$P_1T = (T_x + TP_0) \qquad Q_1T = (T_t + TQ_0). \tag{1.3}$$

The gauge transformation T takes values in $GL(n)$ so that we can use the standard decomposition theorems to factorise T. In particular the Gauss decomposition gives

$$T = \Delta^- \Delta^+ \tag{1.4}$$

where Δ^+ is upper unipotent (ones on the diagonal) and Δ^- is lower triangular. This factorisation implies the existence of an intermediate equation $\{P^1, Q^1\}$ in the sense of

$$(P_0^0, Q_0^0) \xrightarrow{\Delta^+} (P_0^1, Q_0^1) \xrightarrow{\Delta^-} (P_1^0, Q_1^0)$$

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where we have written $\{P^0, Q^0\} \equiv \{P, Q\}$. In all diagrams arrows will indicate maps between the fundamental solutions of the zero-curvature representations. However, unless the components of Δ^+, Δ^- satisfy some non-trivial conditions the equation $\{P^1, Q^1\}$ is empty. The nature of these conditions is revealed in the example below which is based on the AKNS system [3]. Let $E_{ij} = (\delta_{ui}\delta_{jv})_{1\leq u,v\leq n}$ be the $n \times n$ matrix with 1 in the ijth entry and zero elsewhere.

1.1. *The* AKNS *system*

1.1.1. The ABT *for the* AKNS *system.* The AKNS zero-curvature representation involves elements of sl(2, C). Define $\boldsymbol{h} := E_{11} - E_{22}$, $\boldsymbol{e} := E_{12}$, $\boldsymbol{f} := E_{21}$ and adopt the convention

$$\Delta^+(a) := E_{11} + aE_{12} + E_{22} \qquad \Delta^-(b, c, d) := bE_{11} + cE_{21} + dE_{22}$$

then

$$P_i^0(k) = k\boldsymbol{h} + q_i\boldsymbol{e} + r_i\boldsymbol{f} \qquad Q_i^0(k) = A_i(k)\boldsymbol{h} + B_i(k)\boldsymbol{e} + C_i(k)\boldsymbol{f}. \tag{1.5}$$

Assume that $|q_i|, |r_i| \to 0$ as $|x| \to \infty$; then solving (1.3) for T_0^0 linear in k we find that the Bäcklund transformation is given by

$$\begin{aligned} v_1 q_{0,x} - v_2 q_{1,x} &= (q_1 p_2 - q_0 p_1) + (q_1 v_2 + q_0 v_1) J_{10} \\ v_1 r_{1,x} - v_2 r_{0,x} &= (r_1 p_1 - r_0 p_2) - (r_1 v_1 + r_0 v_2) J_{10} \end{aligned} \tag{1.6}$$

where $J_{10} = \int_x^\infty (q_1 r_1 - q_0 r_0)\, dy$ and v_i, p_i are constants. The transformation is defined by

$$\begin{aligned} T_0^0 := {} & [v_1 k + \tfrac{1}{2}(p_1 - v_1 J_{10})] E_{11} + \tfrac{1}{2}(q_0 v_1 - q_1 v_2) E_{12} + \tfrac{1}{2}(r_1 v_1 - r_0 v_2) E_{21} \\ & + [v_2 k + \tfrac{1}{2}(p_2 + v_2 J_{10})] E_{22}. \end{aligned} \tag{1.7}$$

Then

$$\Delta_0^{+0} := \Delta^+(a_0) \qquad \Delta_0^{-0} := \Delta^-(b_0, c_0 d_0) \tag{1.8}$$

are uniquely defined by (1.4).

The transformations (1.6) contain the elementary Bäcklund transformations [4], which are obtained upon restriction,

$$\begin{aligned} & q_{0,x} = \mu_2 q_1 - \mu_1 q_0 - r_1 q_0^2 \mu q^{-1} \qquad r_{1,x} = \mu_1 r_1 - \mu_2 r_0 + r_1^2 q_0 \mu_2^{-1} \\ & \mu_1 = p_1/v_1 \qquad \mu_2 = p_2/v_1 \qquad v_2 = 0 \end{aligned} \tag{1.9}$$

$$\begin{aligned} & q_{1,x} = \lambda_1 q_0 - \lambda_2 q_1 - \lambda_2^{-1} r_0 q_1^2 \qquad r_{0,x} = \lambda_2 r_0 - \lambda_1 r_1 + \lambda_2^{-1} q_1 r_0^2 \\ & \lambda_1 = p_1/v_2 \qquad \lambda_2 = p_2/v_2 \qquad v_1 = 0. \end{aligned} \tag{1.10}$$

Notice in equations (1.6), (1.9) and (1.10) only the x part of the ABT has been written down. The t part differs from equation to equation in the family of solvable equations associated with the AKNS system.

1.1.2. The intermediate equation. Equations (1.3) with T replaced by Δ_0^{+0} and P_1^0, Q_1^0 replaced by P_0^1, Q_0^1 respectively determine $\{P^1, Q^1\}$,

$$\begin{aligned} P_0^1 &= (k + a_0 r_0)\boldsymbol{h} + (-2a_0 k - a_0^2 r_0 + q_0 + a_{0,x})\boldsymbol{e} + r_0\boldsymbol{f} \\ Q_0^1 &= (A_0 + a_0 C_0)\boldsymbol{h} + (a_{0,t} - 2a_0 A_0 + B_0 - a_0^2 C)\boldsymbol{e} + C_0\boldsymbol{f}. \end{aligned} \tag{1.11}$$

The equation $\{P^1, Q^1\}$ is identically zero. Observe that if (P_0^1, Q_0^1) is to define a non-trivial $\{P^1, Q^1\}$ then P_0^1 and Q_0^1 must only be functionals of the variable a_0. Therefore put

$$a_{0,x} - 2a_0 k - a_0^2 r_0 + q_0 = 0 \qquad a_{0,t} - 2a_0 A_0 + B_0 - a_0^2 C_0 = 0. \tag{1.12}$$

Eliminate (r_0, q_0) and their derivatives from these equations and we get the intermediate equation. The first equation in (1.12) is the Miura transformation for the AKNS system. In particular, we have the three cases

(i) $r_0=-1$

(ii) $r_0=\varepsilon q_0=\varepsilon \bar{q}_0$

(iii) $r_0=\varepsilon \bar{q}_0 \qquad \varepsilon=\pm 1$

for which

(i) $q_0=2a_0k-a_0^2-a_{0,x}$

(ii) $q_0=(2a_0k-a_{0,x})/(1-\varepsilon a_0^2)$

(iii) $q_0=(2\varepsilon a_0^2\bar{a}_0\bar{k}+2a_0k-\varepsilon a_0^2\bar{a}_{0,x}-a_{0,x})/(1-a_0^2\bar{a}_0^2)$.

Equation (i) on rescaling $a_0\to k^{-1}a_0$ is the Gardner transformation and (ii), (iii) are the corresponding transformations in the other cases.

2. The family of factorisations associated with the KdV equation

In this section we shall derive a more invariant way of solving the factorisation problem and consider the case (i) for the KdV equation in some detail. This will enable us to see that the method can be extended an arbitrary number of times and consequently defines a new type of hierarchy which can be associated with a given solvable equation.

For the KdV equation

$$q_t+6qq_x+q_{3x}=0 \tag{2.1}$$

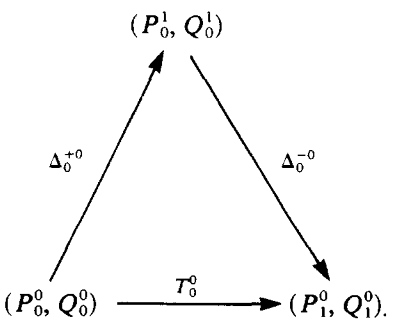
the coefficients of Q_0^0 are

$$A_0=(-4k^3-2kq_0-q_{0,x}) \qquad B_0=(-4k^2q_0-2kq_{0,x}-2q_0^2-q_{0,2x})$$
$$C_0=4k^2+2q_0. \tag{2.2}$$

Then a_0 satisfies the equation

$$a_{0,t}-6a_0^2a_{0,x}+12ka_0a_{0,x}+a_{0,3x}=0. \tag{2.3}$$

We have the following structure:



To check for consistency we need to show that Δ_0^{-0} can be defined so that $T_0^0\equiv\Delta_0^{-0}\circ\Delta_0^{+0}$. From (1.7) we get

$$T_0^0=(k+s_0)E_{11}+\tfrac{1}{2}(q_0+q_1)E_{12}-E_{21}+(-k+s_0)E_{22}$$

where

$$s_0 := \tfrac{1}{2}(\alpha_0 - J_{10}) \qquad J_{10} := -\int_x^\infty (q_1 - q_0)\,\mathrm{d}y$$
$$\alpha_0 = p_1/v_1 \qquad v_1 = -v_2 \qquad p_1 = -p_2. \tag{2.4}$$

The ABT for the KdV (2.1) is

$$(q_1 + q_0)_x = (q_1 - q_0)(\alpha_0 - J_{10}). \tag{2.5}$$

If we use this then we have an alternative expression for s_0. From the definition of s_0 and (2.5) we get

$$2s_0 s_{0,x} = -\tfrac{1}{2}(q_1 - q_0)(\alpha_0 - J_{10}) = -\tfrac{1}{2}(q_0 + q_1)_x \tag{2.6}$$

so that

$$s_0 = \varepsilon[\gamma_0^2 - \tfrac{1}{2}(q_0 + q_1)]^{1/2} \tag{2.7}$$

and γ_0^2 is an arbitrary integration constant.

Therefore for this case we have from (1.4) that the Gauss decomposition of T_0^0 is given by

$$\Delta_0^{+0} = \Delta^+(\tfrac{1}{2}(q_0 + q_1)(s_0 + k)^{-1})$$
$$\Delta_0^{-0} = \Delta^-(s_0 + k, -1, (\gamma_0^2 - k^2)(s_0 + k)^{-1}). \tag{2.8}$$

We now use Δ_0^{+0} and check that $(P_0^1, Q_0^1) \overset{\Delta_0^{-0}}{\to} (P_1^0, Q_1^0)$. Consistency requires that $\varepsilon = 1$ and $\gamma_0^2 = k^2$ so that $\Delta_0^{-0}(k^2 = \gamma_0^2)$ and $T_0^0(k^2 = \gamma_0^2)$ are singular. However (1.3) still defines the ABT for the KdV equation. From (2.7) and (2.8) it follows that

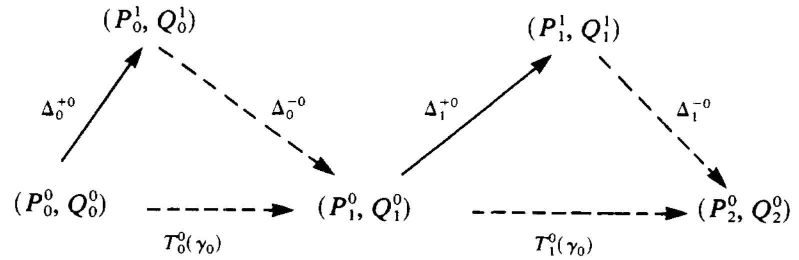
$$a_0 = \gamma_0 - s_0 \qquad a_{0,t} - 6a_0^2 a_{0,x} + 12\gamma_0 a_{0,x} + a_{0,3x} = 0. \tag{2.9}$$

We have chosen $+\gamma_0$ in (2.9), and there is no loss of generality since $\gamma_0 \to -\gamma_0$ gives the other equation.

We also have

$$(P_0^0, Q_0^0) \to (P_0^1, Q_0^1) \qquad q_0 = 2a_0\gamma_0 - a_0^2 - a_{0,x}$$
$$(P_0^1, Q_0^1) \to (P_1^0, Q_1^0) \qquad q_1 = 2a_0\gamma_0 - a_0^2 + a_{0,x}. \tag{2.10}$$

Thus the intermediate equation is associated with the singular transformations $T_0^0(k = \pm\gamma_0)$. Consider the existence of the following transformations:



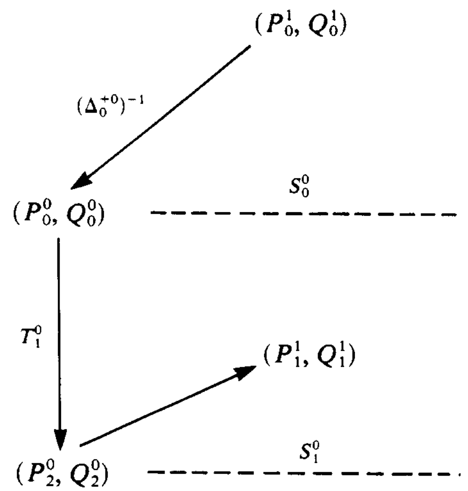
which are to be used to define an ABT for $\{P^1, Q^1\}$. Taking into account equation (2.9) we have the decompositions $\Delta_1^{+0} := \Delta^+(a_1)$, $\Delta_1^{-0} := \Delta^-(2\gamma_0 - a_0, -1, 0)$ from which it appears that we get the pair of equations

$$(a_0 + a_1)_x = [2\gamma_0 - (a_0 + a_1)](a_1 - a_0)$$
$$(a_0 + a_1)_t = [2\gamma_0 - (a_0 + a_1)][6\gamma_0(a_0^2 - a_1^2) - 2(a_0^3 - a_1^3) + (a_0 - a_1)_{2x}]. \tag{2.11}$$

Observe that the x part of this transformation follows directly from (2.10). However, (2.11) is not an ABT for the $\{P^1, Q^1\}$ equation. This can be traced back to the fact that we have moved outside the space of allowable transformations (Δ_0^{-0} is not an element of GL(2)). Notice that in general the Gauss decomposition is then no longer unique. However, it is apparent from the construction that $T_0^0 \in \mathrm{GL}(2)$ for all values of k except $k = \pm\gamma_0$, i.e. the singular transformations are a limiting value of well defined elements of GL(2) with unique Gauss decompositions.

We conclude the following: (i) intermediate equations are determined by the factorisation of singular gauge transformations which are obtained as limiting forms of non-singular gauge transformations; (ii) the singular gauge transformations factorise into a singular and non-singular part which belongs to the gauge group.

We can generate an ABT for (2.3) by using the following diagram (broken lines denote singular transformations):



In the diagram and in the rest of the paper T^j (S^j) will denote a non-singular (singular) transformation. The superscript j will refer to the jth intermediate equation with $j = 0$ denoting the 'seed' equation—the KdV in this case. The non-singular transformation defining the ABT for the first intermediate equation (2.3) is $T_0^1 := \Delta_1^{+0} \circ T_1^0 \circ (\Delta_0^{+0})^{-1}$. Since $\Delta_0^{+0} := \Delta^+(a_0), \Delta_1^{+0} := \Delta^+(a_1)$ and $T_1^0 = (k + s_1)E_{11} + (\gamma_1^2 - s_1^2)E_{12} - E_{21} + (-k + s_1)E_{22}$ where

$$\begin{aligned} &s_1 := [\gamma_1^2 - \tfrac{1}{2}(q_0 - q_2)]^{1/2} \qquad && q_0 = 2ka_0 - a_0^2 - a_{0,x} \\ &q_2 = 2ka_1 - a_1^2 - a_{1,x} && k^2 = \gamma_0^2 \end{aligned} \tag{2.12}$$

equation (1.3) gives for the ABT

$$\begin{aligned} &(a_0 + s_1)_x = 2ka_0 - \gamma_1^2 + s_1^2 - a_0^2 \\ &(s_1 - a_2)_x = \gamma_1^2 - s_1^2 - 2ka_2 + a_2^2 \\ &\gamma_1^2 \neq k^2 = \gamma_0^2. \end{aligned} \tag{2.13}$$

These two relations are consistent since subtracting them results in an expression which is identically satisfied through the definition of s_1. Therefore the x part of the ABT is given by the first equation upon substituting for s_1 and using the Miura transformations defined in (2.12). Explicitly, the ABT (2.13) for equation (2.9) is

$$[\gamma_1^2+\tfrac{1}{2}(a_0^2+a_1^2)+\tfrac{1}{2}(a_0+a_1)_x-k(a_1+a_0)]^{1/2}$$
$$=\gamma_1-\tfrac{1}{2}(a_0-a_1)-\tfrac{1}{2}\int_x^\infty (a_1^2-a_0^2)\,\mathrm{d}y-k\int_x^\infty (a_0-a_1)\,\mathrm{d}y \qquad k^2=\gamma_0^2. \tag{2.14}$$

In particular, if we put $k=0$ then we obtain an ABT for the modified KdV equation (equation (2.3) with $k=0$). Since the modified KdV belongs to the AKNS system it might be thought that the transformation (1.26) can be constructed from the elementary Bäcklund transformations defined in (1.9). However, if we consider the corresponding maps T of the fundamental solutions of the zero-curvature representations, then it is clear from (1.5) and (1.11) that first we have to gauge transform the fundamental solutions corresponding to the solutions a_0 and q_0 of the modified KdV equation in the two different representations. A calculation shows that they are related by a k-dependent (Bäcklund) transformation which is necessarily distinct from the elementary Bäcklund transformations of the AKNS system. Thus special solutions obtained from (2.14) should exist, outside the class of special solutions obtained directly from the AKNS system ($r=\varepsilon q$, $\varepsilon\pm 1$; $\varepsilon=1$ corresponds to the MKdV obtained from (2.3)). We have not investigated the problem of canonical representations in this paper, but if we start with $a_0=0$, equation (2.14) can be explicitly solved in the $k=0$ case to give a singular solution. This is best done by using (2.13)(i) $s_{1,x}-s_1^2+\gamma_1^2=0$, $s_1=-\gamma_1\coth\gamma_1(x+c(t))$ followed by (2.13)(ii), $a_{1,x}+a_1^2=2\gamma_1^2\operatorname{cosech}^2\gamma_1(x+c(t))$ which gives

$$a_1=-2\gamma_1\operatorname{cosech}2\theta+\tanh^2\theta(x-\gamma_1^{-1}\tanh\theta+d)^{-1} \tag{2.15}$$

where $\theta=\gamma_1(x+c(t))$, $c(t)=-4\gamma_1^2 t$ and d is a constant. The function $c(t)$ is determined either directly from the equation or from the t part of the Bäcklund transformation.

The non-singular transformation defining the ABT for the first intermediate equation was shown above to be $T_0^1:=\Delta_1^{+0}\circ T_1^0\circ(\Delta_0^{+0})^{-1}$. The next intermediate equation is therefore obtained by factorising the singular transformation $S_0^1:=\operatorname{sing}T_0^1=\Delta_0^{-1}\circ\Delta_0^{+1}$. This is given by $\Delta_0^{+1}=\Delta^+(f_0)$, $\Delta_0^{-1}=\Delta^-(g_0,-1,h_0)$ with

$$\begin{aligned} &f_0=-a_0+[\gamma_1^2-s_1^2+a_1(s_1-k)](k+s_1-a_1)^{-1}\\ &g_0=(k+s_1-a_1) \qquad h_0=(\gamma_1^2-k^2)(k+s_1-a_1)^{-1} \qquad k^2=\gamma_0^2. \end{aligned} \tag{2.16}$$

Consequently $S_0^1=T_0^1(k^2=\gamma_1^2)$, $k^2=\gamma_0^2$ and so as in the previous case there are two possibilities and we shall restrict ourselves to the case $\gamma_0=k=\gamma_1$. The intermediate equation is then determined from (1.3) using $T=\Delta_0^{+1}$. This gives

$$\begin{aligned} &P_0^2=(\gamma_0-a_0-f_0)\boldsymbol{h}+[f_{0,x}+2f_0(a_0-\gamma_0)+f_0^2]\boldsymbol{e}-\boldsymbol{f}\\ &Q_0^2=[A_0+(a_0+f_0)C_0]\boldsymbol{h}+[f_{0,t}-2f_0(A_0+a_0C_0)-f_0^2C_0]\boldsymbol{e}+C_0\boldsymbol{f}. \end{aligned} \tag{2.17}$$

Then the map $(P_0^2, Q_0^2)\overset{\Delta_0^{-1}}{\to}(P_1^1, Q_1^1)$ requires that

$$f_{0,x}+2f_0(a_0-\gamma_0)+f_0^2=0 \qquad f_{0,t}-2f_0(A_0+a_0C_0)-f_0^2C_0=0. \tag{2.18}$$

The first equation with $f_0=\mathrm{e}^{v_0}$ is

$$a_0=-\tfrac{1}{2}v_{0,x}-\tfrac{1}{2}\mathrm{e}^{v_0}+\gamma_0 \tag{2.19}$$

so that the next intermediate equation $\{P^2, Q^2\}$ is

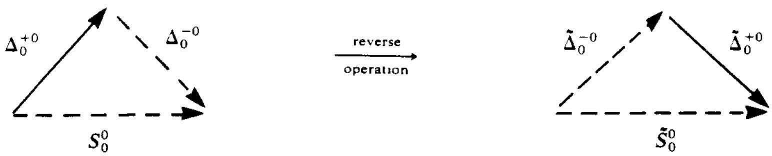
$$v_t + v_{3x} - \tfrac{1}{2}v_x^3 + 6\gamma_0^2 v_x - \tfrac{3}{2}v_x\, \mathrm{e}^{2v} = 0$$

or

$$f_t + f_{3x} - \frac{3f_{2x}f_x}{f} + \frac{3}{2}\left(\frac{f_x^3}{f^2}\right) + 6\gamma_0^2 f_x - \tfrac{3}{2}f_x f^2 = 0. \tag{2.20}$$

The process can be repeated an arbitrary number of times and so defines a new hierarchy which can be associated with the KdV equation. If we let v_j^i denote a jth variable which satisfies the ith intermediate equation starting from v_0^0 then the intermediate equations and their Bäcklund transformations can be derived from figure 1.

The reverse (as opposed to the transpose) diagram arises from the factorisation $T = \tilde{\Delta}^+\tilde{\Delta}^-$ and in this case we get



and the diagram corresponding to figure 1 is easily obtained. The intermediate equations are invariant under this operation.

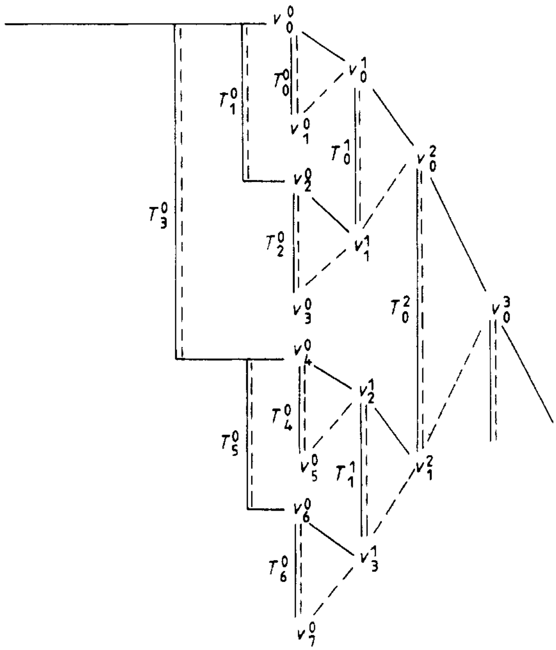


Figure 1. The hierarchy of intermediate equations and Bäcklund transformations associated with the KdV. Moving between variables connected by an oblique line to the right corresponds to a transformation to the next intermediate equation, causing the previous ABT to become singular. This requires that the component of the transformation arising from the ABT of the KdV should be singular. The KdV transformations are the vertical lines in the diagram and a singular transformation is denoted by a broken line.

The intermediate equation $\{P^3, Q^3\}$ is associated with the singular transformation derived from

$$T_0^2 := \Delta_0^{+1} \circ \Delta_4^{+0} \circ T_3^0 \circ (\Delta_0^{+0})^{-1} (\Delta_0^{+1})^{-1}. \tag{2.21}$$

From the diagram it is clear that $S_0^2 = T_0^2(k=\gamma_3)$, $k^2 = \gamma_0^2$, where as usual we only consider one of the possibilities. The associated intermediate equation is given by

$$\begin{aligned} &u_{0,x} - 2u_0(\gamma_0 - a_0 - f_0) + u_0^2 = 0 \\ &u_{0,t} - 2u_0[A_0 + (a_0 + f_0)C_0] - f_0^2 C_0 = 0. \end{aligned} \tag{2.22}$$

The first equation in (2.22) can be resolved to give

$$u_{0,x}/u_0 + u_0 = -f_0 + f_{0,x}/f_0 \tag{2.23}$$

so that in terms of the potential $\phi_0 = \int^x u_0 \, dx$,

$$f_0 = -[\ln(e^{\phi} - \alpha)]_{,x} \tag{2.24}$$

where the function $\alpha(t)$ is arbitrary. Although we can find a Bäcklund transformation and a zero-curvature representation for the new function ϕ the corresponding equation is not very interesting since if we put $\phi = \ln(e^s + \alpha)$ then (2.24) corresponds to the introduction of a potential function. We shall call such transformations trivial.

Proposition 1. In the hierarchy of intermediate equations associated with the Gauss decomposition of gauge transformations of the type considered in this section for the KdV equation $v_t - 6vv_x + v_{3x} = 0$, the only non-trivial equations are

$$\begin{aligned} &v_t - 6v^2 v_x + 12kvv_x + v_{3x} = 0 \\ &v_t + v_{3x} - 3\frac{v_{2x}v_x}{v} + \frac{3}{2}\left(\frac{v_x^3}{v^2}\right) + 6k^2 v_x - \tfrac{3}{2}\, v_x v^2 = 0. \end{aligned}$$

The proof is straightforward since for the nth intermediate equation $P_0^n = (k - \Sigma_{i=0}^n v^{(i)})\boldsymbol{h} - \boldsymbol{f}$ where $v^{(0)} \equiv q_0$, $v^{(1)} \equiv a_0$, $v^{(2)} \equiv f_0$ and the Miura-type transformation defining the nth equation is

$$v_{,x}^{(n)} + 2v^{(n)}\left(\sum_{i=1}^{n-1} v^{(i)} - k\right) + (v^{(n)})^2 = 0.$$

The result is established by induction.

3. Some intermediate equations for the AKNS system

For simplicity we shall only consider two examples of case (ii) given in § 1. Case (iii) is similar but more cumbersome to write out explicitly.

(ii) $$q = \frac{2ak - a_x}{1 - \varepsilon a^2} \qquad \varepsilon = \pm 1.$$

As an example we have for the MKdV $q_t + 6\varepsilon q^2 q_x + q_{3x} = 0$ with $\varepsilon = -1$,

$$\begin{aligned} &A = -4k^3 - 2kq^2 \qquad B = -4k^2 q - 2kq_x - q_{2x} - 2q^3 \\ &C = 4k^2 q - 2kq_x + q_{2x} + 2q^3. \end{aligned} \tag{3.1}$$

Put $b = 2a/(1+a^2) = \sin v$; then we get

$$v_t + v_{3x} + \tfrac{1}{2}v_x^3 + 6k^2 v_x \sin^2 v = 0$$

or

$$b_t + \left(b_{2x} + 3k^2 b^3 + \frac{3}{2}\frac{b^2 b_x^2}{(1-b^2)} \right)_{,x} = 0. \tag{3.2}$$

In [5] this is called the modified-modified KdV equation and a linear deformation problem was directly derived for it in [6]. The sine-Gordon equation

$$\begin{aligned} &u_{xt} = \sin u \qquad q = -\tfrac{1}{2}u_x \qquad A = (1/4k)\cos u \\ &B = (1/4k)\sin u \qquad C = (1/4k)\sin u \end{aligned} \tag{3.3}$$

can be shown to give the equation investigated by Kruskal [7],

$$v_{xt} = (1 - k^2 v_t^2)^{1/2} \sin v \qquad a = \tan \tfrac{1}{2}v. \tag{3.4}$$

The next intermediate equation is determined from the equations

$$\begin{aligned} &a_{1,x} - 2ka_1 - 2a_0 a_1 r_0 - a_1^2 r_0 = 0 \\ &a_{1,t} - 2a_1 A_0 - 2a_1 a_0 C_0 - a_1^2 C_0 = 0. \end{aligned} \tag{3.5}$$

The first equation in (3.5) for case (ii) is

$$a_{1,x} - 2ka_1 - \varepsilon \frac{(2ka_0 - a_{0,x})}{(1 - \varepsilon a_0^2)} a_1(a_1 + 2a_0) = 0 \tag{3.6}$$

so that it follows that the equation for a_1 given by the second relation in (3.5) will still involve a_0. The transformation is still a Bäcklund transformation (this is the situation which usually arises in the theory of Bäcklund transformations [8]), but we shall not consider transformations of this type.

4. Factorisation of *n*-component systems

The following points follow immediately from the previous studies. Let $Y_x = PY$, $Y_t = QY$ define a zero-curvature representation of $\{P, Q\}$; then if there exists a gauge transformation $T(k)$ such that $Y_1 = TY_0$ defines an ABT for $\{P, Q\}$ (i) the intermediate equation is associated with the unique transformation Δ^+ where $S(k) = \operatorname{sing} T(k) = \Delta^-\Delta^+$ and $\det S(k) = 0 = \det \Delta^-$; (ii) Δ^+ factorises, $\Delta^+ = \Pi_{i<j}\,\Delta^+(a_{ij})$ where

$$\Delta^+(a_{ij}) = I + a_{ij}E_{ij}$$

(iii) if $\operatorname{diag}\Delta^- = \Sigma_{j=1}^n a_{j1}E_{j1}$ and $P_0^0 \overset{\Delta^+}{\to} P_0^1$, $Q_0^0 \overset{\Delta^+}{\to} Q_0^1$ then P_0^1 and Q_0^1 are lower triangular.

Point (iii) follows from

$$P_1^0\Delta^- = \Delta_x^- + \Delta^- P_0^1 \tag{4.1}$$

where $P_0^0 \overset{T}{\to} P_0^1$, since under the conditions (iii), $P_1^0\Delta^- = \tilde{\Delta}^-$.

4.1. Two-component systems

As a consequence of point (iii), the Miura transformation can be written directly for all two-component zero-curvature systems. As an example, the Ernst equation [9]

$$2\varepsilon_{x\bar{x}}+\rho^{-1}(\varepsilon_x+\varepsilon_{\bar{x}})=2f^{-1}\varepsilon_x\varepsilon_{\bar{x}} \tag{4.2}$$

where $\varepsilon=f+\mathrm{i}\psi$, $x=\rho+\mathrm{i}z$, is associated with the linear problem $Y_x=PY$, $Y_{\bar{x}}=QY$ for which [10]

$$\begin{aligned}P&=\tfrac{1}{2}f^{-1}(\bar{\varepsilon}_xE_{11}+\gamma^{1/2}\bar{\varepsilon}_xE_{12}+\varepsilon_xE_{21}+\gamma^{1/2}\varepsilon_xE_{22})\\ Q&=\tfrac{1}{2}f^{-1}(\bar{\varepsilon}_{\bar{x}}E_{11}+\gamma^{-1/2}\bar{\varepsilon}_{\bar{x}}E_{12}+\varepsilon_{\bar{x}}E_{21}+\gamma^{-1/2}\varepsilon_{\bar{x}}E_{22})\end{aligned} \tag{4.3}$$

where $\gamma=(1-2\mathrm{i}k\bar{x})/(1+2\mathrm{i}kx)$, $k\in C$.

The Miura transformation is given by

$$\begin{aligned}\varepsilon_x&=\frac{2f}{(\gamma^{1/2}-a)(\bar{\gamma}^{-1/2}-\bar{a})}\left(\frac{(\bar{\gamma}^{-1/2}-\bar{a})\bar{a}a_x-(\gamma^{1/2}-a)\bar{a}_x}{(1-a\bar{a})}\right)\\ \varepsilon_{\bar{x}}&=\frac{2f}{(\bar{\gamma}^{1/2}-\bar{a})(\gamma^{-1/2}-a)}\left(\frac{(\bar{\gamma}^{1/2}-\bar{a})\bar{a}a_{\bar{x}}-(\gamma^{-1/2}-a)\bar{a}_{\bar{x}}}{(1-a\bar{a})}\right).\end{aligned} \tag{4.4}$$

The intermediate equation is obtained from (4.2).

4.2. Three-component systems

A three-component zero-curvature representation can easily be obtained for the scalar Lax equation $L_t=[A,L]$ defined by the L operator

$$L=\partial_x^3+u_1\partial_x+u_0 \tag{4.5a}$$

and with, for example,

$$A=\partial_x^2+\tfrac{2}{3}u_1. \tag{4.5b}$$

The factorisation of this operator through first-order operators has already been investigated in the literature [11, 12]. Our purpose here is to derive the result from the equivalent zero-curvature representation. A straightforward approach gives for $L\psi=k^3\psi$ the system

$$Y_x=[E_{12}+E_{23}+(k^3-u_0)E_{31}-u_1E_{32}]Y \tag{4.6}$$

which does not admit an ABT gauge transformation $T(k)$. Therefore transform (4.6) using

$$W=(-\omega^2k^2E_{11}-\omega kE_{12}-E_{13}+kE_{21}-E_{22}+E_{31})Y \qquad \omega^3=1. \tag{4.7}$$

Equation (4.6) transforms to

$$\begin{aligned}&W_x=P_0^0W\\ &P_0^0=k\omega E_{11}-u_1E_{12}+(u_0+ku_1)E_{13}+E_{21}+\omega^2kE_{22}-E_{32}+kE_{33}.\end{aligned} \tag{4.8}$$

The zero-curvature representation of P_0^1 obtained from the gauge transformation

$$\Delta^+=I+a_1E_{12}+a_2E_{13}+a_3E_{23} \tag{4.9}$$

is

$$P_0^1=(k\omega+a_1)E_{11}+s_2E_{12}+s_3E_{13}+(k\omega^2-a_3-a_1)E_{22}+s_6E_{23}-E_{32}+(k+a_3)E_{23}$$

with

$$\begin{aligned} s_2 &= a_{1,x} + k\omega(\omega-1)a_1 - a_1^2 - a_2 - u_1 \\ s_3 &= u_0 + ku_1 + k(1-\omega)a_2 - a_1a_2 - a_3a_2 + a_{2,x} \\ s_6 &= a_{3,x} + k(1-\omega^2)a_3 - a_2 + a_3^2 + a_1a_3. \end{aligned} \tag{4.10}$$

Observe that to obtain a non-trivial result we can put $s_2=0$, $s_3=0$, $s_6=0$ so that

$$\begin{aligned} a_2 &= a_{3,x} + k(1-\omega^2)a_3 + a_3^2 + a_1a_3 \\ u_1 &= a_{1,x} - a_{3,x} - a_1^2 - a_3^2 - a_1a_3 + k(\omega-1)[\omega a_1 + (\omega+1)a_3] \\ u_0 + ku_1 &= -a_3a_{1,x} - a_{3,2x} - 2a_3a_{3,x} + a_1a_3^2 + a_1^2a_3 \\ &\quad - k(1-\omega)[(2+\omega)a_{3,x} + k(1-\omega^2)a_3 + a_3^2 - \omega a_1a_3]. \end{aligned} \tag{4.11}$$

The last two equations furnish the Miura transformation for this system. It is the same as that derived earlier ($a_1=-y+z$, $a_3=y+z$, $k=0$ in [11]). The next intermediate equation is obtained by factorising the gauge transformation defining the ABT for (P^1, Q^1) where

$$P_0^1 = (k\omega + a_1)E_{11} + E_{21} + (k\omega^2 - a_3)E_{22} - E_{32} + kE_{33}. \tag{4.12}$$

If we use

$$\Delta_0^{+1} = I + b_1E_{12} + b_2E_{13} + b_3E_{23} \tag{4.13}$$

then we find that

$$P_0^2 = (k\omega + a_1 + b_1)E_{11} + E_{21} + (k\omega^2 - a_1 - a_3 - b_1 - b_3)E_{22} - E_{32} + (k + a_3 + b_3)E_{33} \tag{4.14}$$

and the x part of the Bäcklund transformation is given by

$$\begin{aligned} &b_{1,x} - b_2 - b_1a_3 - 2a_1b_1 - b_1^2 + k\omega(\omega-1)b_1 = 0 \\ &b_{3,x} + 2a_3b_3 - b_2 + a_1b_3 + b_3^2 + b_1b_3 + k(1-\omega^2)b_3 = 0 \\ &b_{2,x} + b_2a_3 - a_1b_2 - b_1b_2 + k(1-\omega)b_2 = 0. \end{aligned} \tag{4.15}$$

However, this Bäcklund transformation is of the general type discussed at the end of § 3. A Miura transformation can be obtained from it when $b_2=0$ and then we obtain the following cases.

(i) $a_1 \neq 0$, $a_3 \neq 0$, $a_1 \neq \alpha a_3$:

$$\begin{aligned} a_1 &= \tfrac{1}{3}(2b_{1,x}/b_1 + b_{3,x}/b_3 + b_3 - b_1 - 3\omega k) \\ a_3 &= -\tfrac{1}{3}(b_{1,x}/b_1 + 2b_{3,x}/b_3 + b_1 + 2b_3 + 3k). \end{aligned} \tag{4.16}$$

(ii) $a_1=0$, $a_3 \neq 0$, then $b_1=0$ and

$$a_3 = -\tfrac{1}{2}[b_{3,x}/b_3 + b_3 + k(1-\omega^2)]. \tag{4.17}$$

(iii) $a_3=0$, $a_1 \neq 0$ then $b_1=0$ and

$$a_1 = -[b_{3,x}/b_3 + b_3 + k(1-\omega^2)]. \tag{4.17'}$$

(iv) $a_1 = \alpha a_3$ then $b_3=0$ and

$$a_3 = \left(\frac{1}{1+2\alpha}\right)\left(\frac{b_{1,x}}{b_1} - b_1 + k\omega(\omega-1)\right). \tag{4.18}$$

With the operator A defined in (4.5b) the corresponding system $W_x = P_0^0 W$, $W_t = Q_0^0 W$ defines the Boussinesq equation

$$v_{tt} = -\tfrac{1}{3}(v_{4x} + 2v_x^2) \qquad \text{with} \qquad u_0 = \tfrac{1}{2}v_x + w \qquad u_1 = v. \tag{4.19}$$

The Miura transformation (4.11) then defines the modified Boussinesq equation (i), the modified Sawada-Kotera equation [13], (ii) and (iii), and the modified Kupershmidt equation [12] (iv). The corresponding intermediate equation (4.16)-(4.18) could then be called the modified-modified equation. At the next factorisation, general Bäcklund transformations are obtained.

5. Comments and further extensions

It is clear that this process can be applied to an arbitrary zero-curvature representation of a solvable equation which admits a gauge transformation defining the ABT. In general, it appears that only the first intermediate equation is obtained from a Miura transformation. However, the technique does suggest a classification scheme which is closely related to that used in the classical papers on the subject [14]. In this work ABT and Miura transformations are classified as Bäcklund transformations of type III and II, respectively. Classify families of solvable equations by hierarchical diagrams of the form given in figure 1. The KdV hierarchy has the KdV as the lowest member, but is it possible to continue the diagram backwards, i.e. find a solvable equation which has the KdV as its factorisation? Incidentally, it is worth stressing that figure 1 constructs zero-curvature representations and ABT for all members of a hierarchy.

There are two possible ways of extending the hierarchy. The zero-curvature representations given here are associated with the fundamental representations of sl(n, C) so that we could use any other irreducible representation of the algebra. In this way we can associate an m-component, $m>2$, zero-curvature representation with the two-component AKNS system, for example. This appears to give general Bäcklund transformations on factorisation, rather than Miura transformations. Another possibility is to canonically embed several copies of a particular zero-curvature problem into a zero-curvature representation of higher dimension. We shall consider as an example the KdV zero-curvature representation $P_0^0(k)$, $Q_0^0(k)$ given in § 2. Then $P_0^0(k_1) \oplus P_0^0(k_2)$, $Q_0^0(k_1) \oplus Q_0^0(k_2) \in \mathrm{sl}(4, C)$ (these have the same solution q_0 of the KdV equation). Now factorise the ABT T of the associated larger system (in this case the four-component AKNS system [15]). Let sing $T = \Delta^- \Delta^+$,

$$\Delta^+(a) = I + a_1 E_{12} + a_2 E_{13} + a_3 E_{14} + a_4 E_{23} + a_5 E_{24} + a_6 E_{34} \tag{5.1}$$

and derive all viable restrictions. Observe that the Δ^- map need no longer preserve the block structure of the embedding. All that is required is that T is an ABT for the general system. An example of this type is provided by the non-linear Klein-Gordon equation $\theta_{xt} = \exp 2\theta - \exp(-\theta)$ which is a restriction of the system $\theta_{xt} = \exp 2\theta - \exp(-\theta) \cosh 3\phi$, $\phi_{xt} = \exp(-\theta) \sinh(3\phi)$. The ABT of the system restricts to a Bäcklund transformation of the Dodd-Bullough equation ($\phi = 0$) but it is not an ABT; it generates solutions of the system ($\phi \neq 0$) [16].

From (5.1) we derive the viable restrictions

(i) $a_2 = a_3 = a_4 = a_5 = 0$

$$q = -a_1^2 + 2k_1 a_1 - a_{1,x} \qquad q = -a_{b,x} + 2k_2 a_6 - a_b^2$$

(ii) $a_1 = a_6 = 0$, $a_2 = (k_2 - 2k_1)a_4$

$$q = (1/a^2)[-a_{2,2x} + 2k_1 a_{2,x} + (k_2^2 - k_1^2)a_2]$$
$$a_3 = a_{2,x} + (k_2 - k_1)a_2 \qquad a_5 = (k_2 - 2k_1)^{-1}[a_{2,x} + (2k_2 - k_1)a_2]. \tag{5.2}$$

The first case is the canonical embedding case whereas the second case gives the Miura transformation

$$q = y_x - y^2 + 2\varepsilon k_2 y \qquad a_2 = A\exp\left(-\int^x [y - (k_1 + \varepsilon k_2)]\,\mathrm{d}x\right) \qquad \varepsilon = \pm 1. \tag{5.3}$$

It is clear since $a_1 = a_6 = 0$ in the second case that it is the first case which can again be factorised ($P_0^1 = P_0^1(k_1, a_1) \oplus P_0^1(k_2, a_6)$ where $P_0^1(k, a)$ is the sl(2, C) matrix P_0^1 defined in § 2). Let $\Delta^+(b)$ define the factorisation using the notation introduced in (5.1). Then, as before, there are two viable restrictions:

(i) $b_2 = b_3 = b_4 = b_5 = 0$

$$a_1 = \frac{1}{2b_1}(-b_{1,x} + 2k_1 b_1 - b_1^2) \qquad a_6 = \frac{1}{2b_6}(-b_{6,x} + 2k_2 b_6 - b_6^2)$$

(ii) $b_1 = b_3 = b_5 = b_6 = 0$

$$(a_6 - a_1) = \frac{1}{b_2}[b_{2,x} + b_2(k_2 - k_1)]$$
$$(a_1 + a_6) = \frac{1}{b_4}[b_{4,x} + (k_1 + k_2)b_4 + b_2]. \tag{5.4}$$

Put $b_2 = \exp\int^x z\,\mathrm{d}x$ and to simplify calculations $z = \mathrm{e}^{v}$ then from (5.2) (i) and (5.4) (iia) we get

$$a_1 = \tfrac{1}{2}[-v_x + 2k_1 - \mathrm{e}^{v} - (k_1^2 - k_2^2)\,\mathrm{e}^{-v}]. \tag{5.5}$$

Then since a_1 satisfies (2.3) with $k = k_1$ or by using the t part of the ABT we get

$$v_t + v_{3x} + 6k_1^2 v_x - \tfrac{1}{2}v_x^3 - \tfrac{3}{2}v_x[\mathrm{e}^{v} + (k_1^2 - k_2^2)\bar{\mathrm{e}}^{v}]^2 = 0$$

or

$$z_t + z_{3x} - \frac{3z_x z_{2x}}{z} + \frac{3}{2}\frac{z_x^3}{z^2} - \tfrac{3}{2}z^2 z_x - \tfrac{3}{2}(k_1^2 - k_2^2)\frac{z_x}{z^2} + 3(k_1^2 + k_2^2)z_x = 0. \tag{5.6}$$

Put $b_2 b_4^{-1} = -\mathrm{e}^{-h}$ and obtain an expression for a_1 in terms of h from (5.4) (ii). This expression is (5.5) with $k_2 = k_1$ and consequently h satisfies the first equation in (5.6) with $k_1 = k_2$. Therefore the appropriate parametrisation for $\Delta^+(b)$ is in terms of h and z.

Clearly this process can be extended by further factorisations and by embedding n copies of the KdV zero-curvature representation in the $2n$-component AKNS problem. The process via figure 1 gives zero-curvature representations and ABT for each of the intermediate equations.

In the process of writing this paper, [17] was brought to my notice. In this paper a classification is given of all equations which can be related to the KdV equation by a transformation of the kind $q = F(a, a_x, \ldots, a_{nx})$ where q satisfies the KdV equation and a satisfies an equation of the form $a_t = a_{3x} + f(a, a_x, a_{2x})$. It is straightforward to derive the additional equations listed in this paper by the embedding method of this section.

In [18] deformations of integrable systems are discussed, but this type of classification is different from the one given here which is simpler in that all that is required is that a Bäcklund transformation exists.

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References

[1] Zakharov V E and Shabat A B 1979 *Funct. Anal. Appl.* **13** 160
[2] Zakharov V E and Mikhailov A V 1978 *Sov. Phys.-JETP* **47** 1017
[3] Ablowitz M J, Kaup D J, Newell A C and Segur H 1974 *Stud. Appl. Math.* **53** 249
[4] Calogero F and Degasperis A 1984 *Physica* **14D** 103
[5] Calogero F and Degasperis A 1985 *Inverse Problems* **1** 57
[6] Dodd R K and Fordy A P 1983 *Proc. R. Soc.* A **385** 389
[7] Kruskal M 1974 *Nonlinear Wave Motion* (Providence, RI: AMS Press) p 61
[8] Forsythe A R 1959 *Theory of Differential Equations* vol 6 (New York: Dover)
[9] Ernst F J 1968 *Phys. Rev.* **167** 1175
[10] Neugebauer G 1979 *J. Phys. A: Math. Gen.* **12** L67
[11] Fordy A P and Gibbons J 1981 *J. Math. Phys.* **22** 1170
[12] Kupershmidt B A and Wilson G 1981 *Invent. Math.* **62** 403
[13] Dodd R K and Gibbon J D 1977 *Proc. R. Soc.* A **358** 287
[14] Goursat E 1925 *Le Probleme de Bäcklund* (Paris: Gauthier Villars)
[15] Newell A C 1979 *Proc. R. Soc.* A **365** 283
[16] Fordy A P and Gibbons J 1980 *Commun. Math. Phys.* **77** 21
[17] Drinfel'd V G and Sokolov V V 1986 *Sov. Math.-Dokl.* **32** 361
[18] Kupershmidt B A 1983 *Proc. R. Irish Acad.* A **83** 45